6. LEAST SQUARES ADJUSTMENT OF OBSERVATIONS ONLY

In Chapter 2 the least squares technique of *adjustment of indirect observations* was introduced using the example of fitting a straight line through a series of data points. The "observations" in this example were the x,y coordinates that were indirect measurements of the unknown parameters m and c, the slope and intercept of the line on the y-axis respectively. Subsequent examples of curve fitting (parabola and ellipse) demonstrated this technique and in Chapter 4 adjustment of indirect observations was applied to a level network. An alternative to this technique, known as *least squares adjustment of observations only,* will be introduced in this chapter using the level network example of Chapter 4.

6.1. Adjustment of a Level Network using Least Squares Adjustment of Observations Only

Figure 6.1 shows a diagram of a level network of height differences observed between the fixed stations A (RL 102.440 m) and B (RL 104.565 m) and "floating" stations X, Y and Z whose Reduced Levels (RL's) are unknown. The arrows on the diagram indicate the direction of rise. The Table of Height differences shows the height difference for each line of the network and the distance (in kilometers) of each level run. The height differences can be considered as independent (uncorrelated) and of unequal precision, where the weights of the height differences are defined as being inversely proportional to the distances in kilometres (see Chapter 3, Section 3.5.2)

Line	Height Diff	Dist (km)
1	6.345	1.7
2	4.235	2.5
3	3.060	1.0
4	0.920	3.8
5	3.895	1.7
6	2.410	1.2
7	4.820	1.5

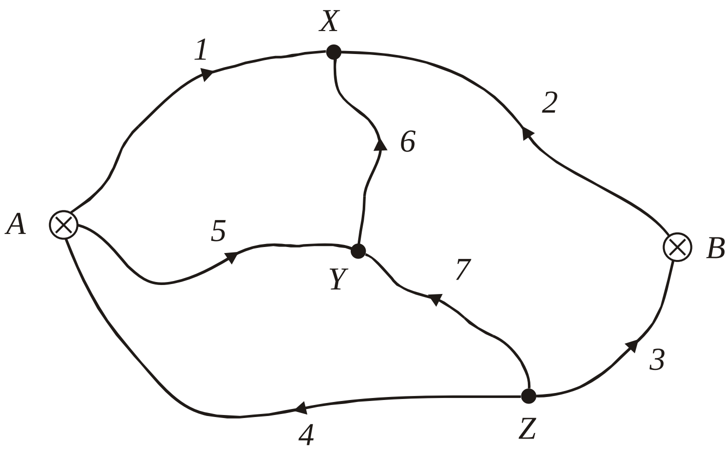

Figure 6.1 Level network

The measured height differences do not accord with the simple principle that they should sum to zero around a "closed loop", i.e., there are misclosures. For example:

in the loop *AXYA* $\Delta H_1 - \Delta H_6 - \Delta H_5 = +0.040$ m

in the loop *XBZYX* $-\Delta H_2 - \Delta H_3 + \Delta H_7 + \Delta H_6 = -0.065$ m

in the loop *AYZA* $\Delta H_5 - \Delta H_7 + \Delta H_4 = -0.005$ m

Hence it is required to determine the adjusted height differences (that will sum to zero) and the RL's of *X, Y* and *Z.*

There are $n = 7$ observations (the measured height differences) and a minimum of $n_0 = 3$ observations are required to fix the RL's of *X, Y* and *Z*. Hence there are $r = n - n_0 = 4$ redundant measurements, which equals the number of independent condition equations. Denoting the observed height differences as $l_1,\ l_2$ etc, residuals as $v_1,\ v_2$ etc and the RL's of *A* and *B* as *A* and *B,* these condition equations are

$$\begin{aligned} (l_1+v_1)-(l_6+v_6)-(l_5+v_5) &= 0 \\ -(l_2+v_2)-(l_3+v_3)+(l_7+v_7)+(l_6+v_6) &= 0 \\ (l_5+v_5)-(l_7+v_7)+(l_4+v_4) &= 0 \\ (l_1+v_1)-(l_2+v_2) &= B-A \end{aligned} \tag{6.1}$$

The first 3 equations of (6.1) are the loop closure conditions and the last equation is a condition linking the RL's of *A* and *B.*

Since the measurements are of unequal precision, there is an associated weight w_k with each observation and the application of the least squares principle calls for the minimization of the least squares function φ as

$$\varphi = \text{ the sum of the weighted squared residuals} = \sum_{k=1}^{n} w_k v_k^2 \tag{6.2}$$

or
$$\varphi = w_1 v_1^2 + w_2 v_2^2 + w_3 v_3^2 + \cdots + w_7 v_7^2$$

Considering equation (6.1) it is clear that separate expressions for residuals cannot be derived and substituted into φ, as was possible in the technique for adjustment of indirect observations (see Chapter 2). Therefore another approach is needed to ensure that φ is a minimum as well as satisfying equations (6.1). This is accomplished by using a method of function minimization developed by Lagrange[1] and set out in the following manner.

(i) Gather the terms in equations (6.1) together

$$
\begin{aligned}
v_1 - v_6 - v_5 &= 0 - \left(l_1 - l_6 - l_5\right) &&= f_1 \\
-v_2 - v_3 + v_7 + v_6 &= 0 - \left(-l_2 - l_3 + l_7 + l_6\right) &&= f_2 \\
v_5 - v_7 + v_4 &= 0 - \left(l_5 - l_7 + l_4\right) &&= f_3 \\
v_1 - v_2 &= \left(B - A\right) - \left(l_1 - l_2\right) &&= f_4
\end{aligned}
\tag{6.3}
$$

(ii) Rewrite equations (6.3) in normal form (zero on the right-hand-side)

$$
\begin{aligned}
v_1 - v_6 - v_5 - f_1 &= 0 \\
-v_2 - v_3 + v_7 + v_6 - f_2 &= 0 \\
v_5 - v_7 + v_4 - f_3 &= 0 \\
v_1 - v_2 - f_4 &= 0
\end{aligned}
\tag{6.4}
$$

(iii) Now form an augmented function φ' of the form

$$
\begin{aligned}
\varphi' = {} & w_1 v_1^2 + w_2 v_2^2 + w_3 v_3^2 + \cdots + w_7 v_7^2 \\
& -2k_1\left(v_1 - v_6 - v_5 - f_1\right) - 2k_2\left(-v_2 - v_3 + v_7 + v_6 - f_2\right) \\
& -2k_3\left(v_5 - v_7 + v_4 - f_3\right) - 2k_4\left(v_1 - v_2 - f_4\right)
\end{aligned}
\tag{6.5}
$$

where $k_1,\ k_2,\ k_3$ and k_4 are *Lagrange multipliers* and there are as many multipliers as there are conditions. The introduction of -2 preceding each multiplier is for convenience only. Inspection of equations (6.5), (6.4) and (6.2) show that φ and φ' are equal since the additional terms in φ' equate to zero.

(iv) The unknowns in equation (6.5) are the residuals $v_1, v_2, \ldots, v_7$ and the Lagrange multipliers $k_1,\ k_2,\ k_3$ and k_4, and so for φ' to be a minimum, the partial derivatives of φ' with respect to each of the unknowns must be zero. Setting the

---

[1] Joseph Louis LAGRANGE (1713-1813), a great French mathematician whose major work was in the calculus of variation, celestial and general mechanics, differential equations and algebra. Lagrange spent 20 years of his life in Prussia and then returned to Paris where his masterpiece, *Mécanique analytique,* published in 1788, formalized much of Newton's work on calculus.

partial derivatives of φ' with respect to the residuals leads to the following equations

$$
\begin{aligned}
\frac{\partial\varphi'}{\partial v_1} &= 2w_1v_1 - 2k_1 - 2k_4 = 0 \qquad \text{or} \qquad v_1 = \frac{1}{w_1}\left(k_1 + k_4\right) \\
\frac{\partial\varphi'}{\partial v_2} &= 2w_2v_2 + 2k_2 + 2k_4 = 0 \qquad \text{or} \qquad v_2 = \frac{1}{w_2}\left(-k_2 - k_4\right) \\
\frac{\partial\varphi'}{\partial v_3} &= 2w_3v_3 + 2k_2 = 0 \qquad \text{or} \qquad v_3 = \frac{1}{w_3}\left(-k_2\right) \\
\frac{\partial\varphi'}{\partial v_4} &= 2w_4v_4 - 2k_3 = 0 \qquad \text{or} \qquad v_4 = \frac{1}{w_4}k_3 \\
\frac{\partial\varphi'}{\partial v_5} &= 2w_5v_5 + 2k_1 - 2k_3 = 0 \qquad \text{or} \qquad v_5 = \frac{1}{w_5}\left(-k_1 + k_3\right) \\
\frac{\partial\varphi'}{\partial v_6} &= 2w_6v_6 + 2k_1 - 2k_2 = 0 \qquad \text{or} \qquad v_6 = \frac{1}{w_6}\left(-k_1 + k_2\right) \\
\frac{\partial\varphi'}{\partial v_7} &= 2w_7v_7 - 2k_2 + 2k_3 = 0 \qquad \text{or} \qquad v_7 = \frac{1}{w_7}\left(k_2 - k_3\right)
\end{aligned}
\tag{6.6}
$$

and when φ' is differentiated with respect to the Lagrange multipliers and equated to zero

$$
\begin{aligned}
\frac{\partial\varphi'}{\partial k_1} &= -2\left(v_1 - v_6 - v_5 - f_1\right) = 0 \qquad \text{or} \qquad v_1 - v_6 - v_5 = f_1 \\
\frac{\partial\varphi'}{\partial k_2} &= -2\left(-v_2 - v_3 + v_7 + v_6 - f_2\right) = 0 \qquad \text{or} \qquad -v_2 - v_3 + v_7 + v_6 = f_2 \\
\frac{\partial\varphi'}{\partial k_3} &= -2\left(v_5 - v_7 + v_4 - f_3\right) = 0 \qquad \text{or} \qquad v_5 - v_7 + v_4 = f_3 \\
\frac{\partial\varphi'}{\partial k_4} &= -2\left(v_1 - v_2 - f_4\right) = 0 \qquad \text{or} \qquad v_1 - v_2 = f_4
\end{aligned}
\tag{6.7}
$$

the original condition equations (6.4) result. This demonstrates that the introduction of Lagrange multipliers ensures that the conditions will be satisfied when φ' is minimized.

(v) Now, substituting equations (6.6) into (6.7) gives four normal equations

$$
\begin{aligned}
\left(\frac{1}{w_1}+\frac{1}{w_6}+\frac{1}{w_5}\right)k_1-\frac{1}{w_6}k_2-\frac{1}{w_5}k_3+\frac{1}{w_1}k_4 &= f_1\\
-\frac{1}{w_6}k_1+\left(\frac{1}{w_2}+\frac{1}{w_3}+\frac{1}{w_6}+\frac{1}{w_7}\right)k_2-\frac{1}{w_7}k_3+\frac{1}{w_2}k_4 &= f_2\\
-\frac{1}{w_5}k_1-\frac{1}{w_7}k_2+\left(\frac{1}{w_4}+\frac{1}{w_5}+\frac{1}{w_7}\right)k_3 &= f_3\\
\frac{1}{w_1}k_1+\frac{1}{w_2}k_2+\left(\frac{1}{w_1}+\frac{1}{w_2}\right)k_4 &= f_4
\end{aligned}
\tag{6.8}
$$

Equations (6.8) can be solved to give the Lagrange multipliers $k_1,\ k_2,\ k_3$ and k_4, which can be substituted back into equations (6.6) to give the residuals $v_1, v_2, \ldots, v_7$. Note that the coefficient terms $\frac{1}{w_k}$ in equations (6.8) are known as weight reciprocals and in the case of levelling are simply the distances of the level runs in kilometres.

Using the data from Figure 6.1 the weight reciprocals are the distances (in kilometres)

$$\frac{1}{w_k}=\{1.7 \quad 2.5 \quad 1 \quad 3.8 \quad 1.7 \quad 1.2 \quad 1.5\}$$

the numeric terms *f* are given by equations (6.3)

$$
\begin{aligned}
f_1 &= \qquad -(l_1-l_6-l_5) &&= -0.040 \text{ m}\\
f_2 &= -(-l_2-l_3+l_7+l_6) &&= 0.065 \text{ m}\\
f_3 &= \qquad -(l_5-l_7+l_4) &&= 0.005 \text{ m}\\
f_4 &= (B-A)-(l_1-l_2) &&= 0.015 \text{ m}
\end{aligned}
$$

and the normal equations (in matrix form) are

$$
\begin{bmatrix} 4.6 & -1.2 & -1.7 & 1.7\\ -1.2 & 6.2 & -1.5 & 2.5\\ -1.7 & -1.5 & 7.0 & 0\\ 1.7 & 2.5 & 0 & 4.2 \end{bmatrix}
\begin{bmatrix} k_1\\ k_2\\ k_3\\ k_4 \end{bmatrix}
=
\begin{bmatrix} -0.040\\ 0.065\\ 0.005\\ 0.015 \end{bmatrix}
\tag{6.9}
$$

The solution of equations (6.9) for the Lagrange multipliers gives

$$k_1=-0.005700,\quad k_2=0.009671,\quad k_3=0.001402,\quad k_4=0.000122$$

Substituting these values (k_1, k_2, k_3 and k_4) together with the weight reciprocals $\frac{1}{w_k}$ into equations (6.6) gives the residuals $v_1, v_2, \ldots, v_7$. The height differences, residuals and the adjusted height differences (observed value + residual) of the level network are shown below.

Line	Observed ΔH	Residual v	Adjusted ΔH
1	6.345	-0.0095	6.336
2	4.235	-0.0245	4.211
3	3.060	-0.0097	3.050
4	0.920	0.0053	0.925
5	3.895	0.0121	3.907
6	2.410	0.0184	2.428
7	4.820	0.0124	4.832

These are identical results to those obtained by least squares adjustment of indirect observations set out in Chapter 4.

6.2. Some Comments on the Two Applications of the Method of Least Squares

1. The method of least squares has been applied in two examples:

   (a) determining the parameters of a "line of best fit" through a number of data points (see Chapter 2) and

   (b) determining the adjusted height differences in a level network.

2. Consider the first example: the *line of best fit.*

   - A mathematical model (equation) was established linking observations, residuals (corrections) and unknown parameters.

- For n observations, there is a minimum number n_0 required to determine the u unknown parameters. In this case $n_0 = u$ and the number of redundant observations is $r = n - n_0$
- An equation was written for each observation, i.e., there were n observation equations. The observation equations were recast as residual equations.
- Since there were n equations in u unknowns ($n \rangle u$) there is no unique solution and the least squares principle was used to determine the u normal equations from which the best estimates of the u unknown parameters were calculated.

This technique of least squares "adjustment" is known by various names, some of which are

parametric least squares,
least squares adjustment by observation equations,
least squares adjustment by residual equations, and
least squares adjustment of indirect observations.

The last of these is perhaps the most explicit since each observation is in fact an indirect measurement of the unknown parameters. Least squares adjustment of indirect observations is the name adopted for this technique by Mikhail (1976) and Mikhail & Gracie (1981) and will be used in these notes.

3. Consider the second example: the *level network.*

- A relationship or condition that the observations (and residuals) must satisfy was established. In this case, the condition to be satisfied was that observed height differences (plus some unknown corrections or residuals) should sum to zero around a closed level loop.
- The minimum number of observations n_0 required to fix the heights of *X, Y* and *Z* and satisfy the condition between the fixed points *A* and *B* was determined giving the number of independent condition equations equal to the number of redundant observations $r = n - n_0$.
- There were r equations in n unknown residuals, and since $r = n - n_0$ was less than n, there was no unique solution for the residuals. The least squares principle was

used to determine a set of r normal equations, which were solved for r Lagrange multipliers which in turn, were used to obtain the n residuals.

- The residuals were added to the observations to obtain the adjusted observations which were then used to determine the heights of points X, Y and Z.

This technique of least squares "adjustment" is known by various names, two of which are

least squares adjustment by condition equations, and
least squares adjustment of observations only.

The second of these is the more explicit since equations involve only observations. No parameters are used. Least squares adjustment of observations only is the name adopted for this technique by Mikhail (1976) and Mikhail & Gracie (1981) and will be used in these notes.

It should be noted that in practice, the method of adjustment of observations only is seldom employed, owing to the difficulty of determining the independent condition equations required as a starting point. This contrasts with the relative ease of the technique of adjustment of indirect observations, where every observation yields an equation of fixed form. Computer solutions of least squares problems almost invariably use the technique of adjustment of indirect observations.

6.2.1. A Note on Independent Condition Equations.

Consider the level network shown in Figure 6.2. The RL of A is known and the RL's of B, C and D are to be determined from the observed height differences. The arrows on the diagram indicate the direction of rise.

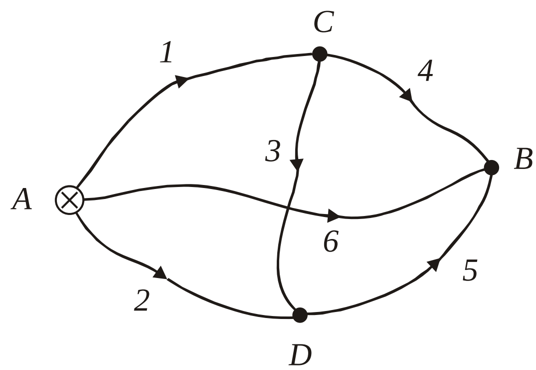

Figure 6.2 Level network

There are $n = 6$ observations with a minimum of $n_0 = 3$ required to fix the RL's of *B, C* and *D* with respect to *A*. Hence there are $r = n - n_0 = 3$ redundant measurements, which equal the number of independent condition equations. Omitting the residuals, these equations are

$$\begin{aligned} l_1 + l_3 - l_2 &= 0 \\ l_4 - l_5 - l_3 &= 0 \\ l_1 + l_4 - l_6 &= 0 \end{aligned} \tag{6.10}$$

Alternatively, here is another set of independent condition equations

$$\begin{aligned} l_1 + l_3 + l_5 - l_6 &= 0 \\ l_1 + l_4 - l_6 &= 0 \\ l_1 + l_4 - l_5 - l_2 &= 0 \end{aligned} \tag{6.11}$$

But, here is a further set of condition equations, which are not independent

$$\begin{aligned} l_1 + l_3 - l_2 &= 0 \\ l_4 - l_5 - l_3 &= 0 \\ l_1 + l_4 - l_5 - l_2 &= 0 \end{aligned} \tag{6.12}$$

where the third equation of (6.12) is obtained by adding the first two.

Care needs to be taken in determining independent equations and it is easy to see that this could become quite difficult as the complexity of the adjustment problem increases.

6.3. Matrix Methods and Least Squares Adjustment of Observations Only

Matrix methods may be used to develop standard equations and solutions for this technique of least squares adjustment.

Consider again the example of the *level net* shown in Figure 6.1. The independent condition equations, (reflecting the fact that height differences around closed level loops should sum to zero and the condition between the known RL's of *A* and *B*), are

$$\begin{aligned} (l_1 + v_1) - (l_6 + v_6) - (l_5 + v_5) &= 0 \\ -(l_2 + v_2) - (l_3 + v_3) + (l_7 + v_7) + (l_6 + v_6) &= 0 \\ (l_5 + v_5) - (l_7 + v_7) + (l_4 + v_4) &= 0 \\ (l_1 + v_1) - (l_2 + v_2) &= B - A \end{aligned} \tag{6.13}$$

These equations could be expressed in matrix form as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} l_1 + v_1 \\ l_2 + v_2 \\ l_3 + v_3 \\ l_4 + v_4 \\ l_5 + v_5 \\ l_6 + v_6 \\ l_7 + v_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ B - A \end{bmatrix} \tag{6.14}$$

or

$$\mathbf{Al} + \mathbf{Av} = \mathbf{d} \tag{6.15}$$

which can be written as

$$\mathbf{Av} = \mathbf{f} \tag{6.16}$$

where

$$\mathbf{f} = \mathbf{d} - \mathbf{Al} \tag{6.17}$$

and

n	is the number of measurements or observations,
n_0	is the minimum number of observations required,
$r = n - n_0$	is the number of redundant observations (equal to the number of condition equations,
$\mathbf{v}$	is an $(n,1)$ vector of residuals,
$\mathbf{l}$	is the $(n,1)$ vector of observations,
$\mathbf{A}$	is an (r,n) matrix of coefficients,
$\mathbf{f}$	is an $(r,1)$ vector of numeric terms derived from the observations,
$\mathbf{d}$	is an $(r,1)$ vector of constants. Note that in many least squares problems the vector **d** is zero.

Now if each observation has an *a priori* estimate of its variance then the (n,n) weight matrix of the observations **W** is known and the least squares function φ is

$$\varphi = \text{ the sum of the weighted squared residuals} = \sum_{k=1}^{n} w_k v_k^2$$

In matrix form, the least squares function is expressed as

$$\varphi = \mathbf{v}^T \mathbf{W} \mathbf{v} \tag{6.18}$$

Now φ is the function to be minimised <u>but</u> with the constraints imposed by the condition equations (6.16). This is achieved by adding an $(r,1)$ vector of *Lagrange multipliers* $\mathbf{k}$ and forming a new function φ'.

$$\varphi' = \mathbf{v}^T\mathbf{W}\mathbf{v} - 2\mathbf{k}^T(\mathbf{A}\mathbf{v} - \mathbf{f}) \tag{6.19}$$

Note that the second term of (6.19) equals zero, since $\mathbf{A}\mathbf{v} - \mathbf{f} = \mathbf{0}$.

Minimising φ' is achieved by differentiating with respect to the unknowns, $\mathbf{v}$ and $\mathbf{k}$ and equating these differentials to zero

$$\frac{\partial\varphi'}{\partial\mathbf{k}} = -2\mathbf{v}^T\mathbf{A}^T + 2\mathbf{f}^T = \mathbf{0}^T \tag{6.20}$$

$$\frac{\partial\varphi'}{\partial\mathbf{v}} = 2\mathbf{v}^T\mathbf{W} - 2\mathbf{k}^T\mathbf{A} = \mathbf{0}^T \tag{6.21}$$

Dividing by two, re-arranging and transposing equations (6.20) and (6.21) gives

$$\mathbf{A}\mathbf{v} = \mathbf{f} \tag{6.22}$$

$$\mathbf{W}\mathbf{v} - \mathbf{A}^T\mathbf{k} = \mathbf{0} \tag{6.23}$$

Note that equations (6.22) are the original condition equations and also that $\mathbf{W} = \mathbf{W}^T$ due to symmetry.

From (6.23), the $(n,1)$ vector of residuals $\mathbf{v}$ is

$$\mathbf{v} = \mathbf{W}^{-1}\mathbf{A}^T\mathbf{k} = \mathbf{Q}\mathbf{A}^T\mathbf{k} \tag{6.24}$$

which, when substituted into (6.22), gives

$$\mathbf{A}\left(\mathbf{Q}\mathbf{A}^T\mathbf{k}\right) = \left(\mathbf{A}\mathbf{Q}\mathbf{A}^T\right)\mathbf{k} = \mathbf{f} \tag{6.25}$$

The matrix $\mathbf{A}\mathbf{Q}\mathbf{A}^T$ is symmetric and of order (r,r) and equations (6.25) are often termed the *normal equations.* The solution of the $(r,1)$ vector of *Lagrange multipliers* $\mathbf{k}$ is

$$\mathbf{k} = \left(\mathbf{A}\mathbf{Q}\mathbf{A}^T\right)^{-1}\mathbf{f} \tag{6.26}$$

Now the term $\mathbf{A}\mathbf{Q}\mathbf{A}^T$ in equations (6.25) and (6.26) can be "simplified" if an <u>equivalent</u> set of observations $\mathbf{l}_e$ is considered, i.e.,

$$\mathbf{l}_e = \mathbf{A}\mathbf{l} \tag{6.27}$$

Applying the general law of propagation of variances (cofactors) to (6.27) gives

$$\mathbf{Q}_e = \mathbf{A}\mathbf{Q}_{ll}\mathbf{A}^T = \mathbf{A}\mathbf{Q}\mathbf{A}^T \tag{6.28}$$

and
$$\mathbf{W}_e = \mathbf{Q}_e^{-1} = \left(\mathbf{A}\mathbf{Q}\mathbf{A}^T\right)^{-1} \tag{6.29}$$

Substituting (6.29) into (6.26) gives another expression for **k**

$$\mathbf{k} = \mathbf{Q}_e^{-1}\mathbf{f} = \mathbf{W}_e\mathbf{f} \tag{6.30}$$

After computing **k** from either (6.26) or (6.30) the residuals **v** are computed from (6.24) and the vector of adjusted observations $\hat{\mathbf{l}}$ is given by

$$\hat{\mathbf{l}} = \mathbf{l} + \mathbf{v} \tag{6.31}$$

<u>This is the standard matrix solution for least squares adjustment of observations only.</u>

6.4. Propagation of Variances for Least Squares Adjustment of Observations Only

In this technique of least squares adjustment, the condition equations in matrix form are

$$\mathbf{A}\mathbf{v} = \mathbf{f} \tag{6.32}$$

with
$$\mathbf{f} = \mathbf{d} - \mathbf{A}\mathbf{l} \tag{6.33}$$

Similarly to Chapter 5, equation (6.33) can be expressed in a form similar to equation (3.23) and the general law of propagation of variances applied to give the cofactor matrix of the numeric terms **f**.

$$\mathbf{f} = -\mathbf{A}\mathbf{l} + \mathbf{d}$$

and
$$\mathbf{Q}_{ff} = (-\mathbf{A})\mathbf{Q}(-\mathbf{A})^T = \mathbf{A}\mathbf{Q}\mathbf{A}^T = \mathbf{Q}_e \tag{6.34}$$

Thus the cofactor matrix of ***f*** is also the cofactor matrix of an <u>equivalent</u> set of observations.

The solution "steps" in the least squares adjustment of observations only are set out above and restated as

$$
\begin{aligned}
\mathbf{Q}_e &= \mathbf{A}\mathbf{Q}\mathbf{A}^T \\
\mathbf{W}_e &= \mathbf{Q}_e^{-1} \\
\mathbf{k} &= \mathbf{W}_e\,\mathbf{f} \\
\mathbf{v} &= \mathbf{Q}\mathbf{A}^T\,\mathbf{k} \\
\hat{\mathbf{l}} &= \mathbf{l} + \mathbf{v}
\end{aligned}
$$

Applying the law of propagation of variances (remembering that cofactor and weight matrices are symmetric) gives the following cofactor matrices

$$\mathbf{Q}_{kk} = (\mathbf{W}_e)\mathbf{Q}_{ff}(\mathbf{W}_e)^T = \mathbf{W}_e \tag{6.35}$$

$$\mathbf{Q}_{vv} = (\mathbf{Q}\mathbf{A}^T)\mathbf{Q}_{kk}(\mathbf{Q}\mathbf{A}^T)^T = \mathbf{Q}\mathbf{A}^T\mathbf{W}_e\mathbf{A}\mathbf{Q} \tag{6.36}$$

and

$$
\begin{aligned}
\hat{\mathbf{l}} &= \mathbf{l} + \mathbf{v} \\
&= \mathbf{l} + \mathbf{Q}\mathbf{A}^T\mathbf{k} \\
&= \mathbf{l} + \mathbf{Q}\mathbf{A}^T\mathbf{W}_e\mathbf{f} \\
&= \mathbf{l} + \mathbf{Q}\mathbf{A}^T\mathbf{W}_e\left(\mathbf{d} - \mathbf{A}\mathbf{l}\right)
\end{aligned}
$$

from which follows

$$\hat{\mathbf{l}} = \left(\mathbf{I} - \mathbf{Q}\mathbf{A}^T\mathbf{W}_e\mathbf{A}\right)\mathbf{l} + \mathbf{Q}\mathbf{A}^T\mathbf{W}_e\mathbf{d} \tag{6.37}$$

Applying the law of propagation of variances to (6.37) gives

$$\mathbf{Q}_{\hat{l}\hat{l}} = \left(\mathbf{I} - \mathbf{Q}\mathbf{A}^T\mathbf{W}_e\mathbf{A}\right)\mathbf{Q}\left(\mathbf{I} - \mathbf{Q}\mathbf{A}^T\mathbf{W}_e\mathbf{A}\right)^T$$

which reduces to

$$\mathbf{Q}_{\hat{l}\hat{l}} = \mathbf{Q} - \mathbf{Q}\mathbf{A}^T\mathbf{W}_e\mathbf{A}\mathbf{Q} = \mathbf{Q} - \mathbf{Q}_{vv} \tag{6.38}$$

Variance-covariance matrices for $\mathbf{k}$, $\mathbf{v}$ and $\hat{\mathbf{l}}$ are obtained by multiplying the cofactor matrix by the variance factor σ_0^2 - see equation (2.32).

The *a priori* estimate of the variance factor may be computed from

$$\hat{\sigma}_0^2 = \frac{\mathbf{v}^T\mathbf{W}\mathbf{v}}{r} \tag{6.39}$$

where

$\mathbf{v}^T\mathbf{W}\mathbf{v}$	is the quadratic form, and
r	is the degrees of freedom.

A derivation of equation (6.39) is given in Chapter 5. The quadratic form $\mathbf{v}^T\mathbf{W}\mathbf{v}$ may be computed in the following manner.

Remembering, for the method of observations only, the following matrix equations

$$\begin{aligned}\mathbf{Q}_e &= \mathbf{A}\mathbf{Q}\mathbf{A}^T\\ \mathbf{W}_e &= \mathbf{Q}_e^{-1}\\ \mathbf{k} &= \mathbf{W}_e\mathbf{f}\\ \mathbf{v} &= \mathbf{Q}\mathbf{A}^T\mathbf{k}\end{aligned}$$

then

$$\begin{aligned}\mathbf{v}^T\mathbf{W}\mathbf{v} &= \left(\mathbf{Q}\mathbf{A}^T\mathbf{k}\right)^T\mathbf{W}\left(\mathbf{Q}\mathbf{A}^T\mathbf{k}\right)\\ &= \mathbf{k}^T\mathbf{A}\mathbf{Q}\mathbf{W}\mathbf{Q}\mathbf{A}^T\mathbf{k}\\ &= \mathbf{k}^T\mathbf{A}\mathbf{Q}\mathbf{A}^T\mathbf{k}\\ &= \mathbf{k}^T\mathbf{Q}_e\mathbf{k}\\ &= \mathbf{k}^T\mathbf{W}_e^{-1}\mathbf{k}\\ &= \mathbf{f}^T\mathbf{W}_e\mathbf{W}_e^{-1}\mathbf{k}\end{aligned}$$

and

$$\mathbf{v}^T\mathbf{W}\mathbf{v} = \mathbf{f}^T\mathbf{k} \tag{6.40}$$

6.5. Adjustment of a Single Closed Traverse using the method of Least Squares Adjustment of Observations Only

The basic component of many surveys is a traverse whose bearings have been determined by theodolite or total station observations and distances measured by EDM. If careful observations are made with well maintained equipment, the measurements are usually free of systematic errors and mistakes and the surveyor is left with small random errors which, in the case of a closed traverse, reveal themselves as angular and linear misclosures. If the misclosures are within acceptable limits, it is standard practice to remove the misclosures by adjusting the original observations to make the traverse a mathematically correct figure. In this section, only single closed traverses are considered and such traverses may begin and end at different fixed points or close back on the starting point. Traverse networks, consisting of two or more single traverses with common junction points, are not considered here; such networks are usually adjusted by a method commonly known as *Variation of Coordinates,* based on Least Squares Adjustment of Indirect Observations.

6.5.1. Some single traverse adjustment methods and their deficiencies

A traverse adjustment method should be based on sound mathematical principles related to the measurement techniques with due allowance made for independence (or dependence) of those measurements and also allow for differing measurement precisions.

Bowditch's Rule and the *Transit Rule,* both of which adjust lengths and bearings of traverse lines and *Crandall's method,* which adjusts the lengths only of the traverse lines, are three popular adjustment methods that fail to meet the general guidelines above. Although Crandall's method, which is explained in detail in later sections, does have mathematical rigour if it assumed that the bearings of a traverse close and require no further adjustment.

Bowditch's Rule and the Transit Rule for adjusting single traverses are explained below by applying the rules to adjust a four-sided polygon having an unusually large misclose. The polygon, shown in Figure 6.3, does not reflect the usual misclosures associated with traverses using modern surveying equipment.

Bowditch's Rule

Nathaniel Bowditch (1773-1838) was an American mathematician and astronomer (see citation below). In 1808, in response to a prize offered by a correspondent in *The Analyst*[2], Bowditch put forward a method of adjusting the misclose in a chain and compass survey (bearings measured by magnetic compass and distances measured by surveyor's chain). His method of adjustment was simple and became widely used. It is still used today for the adjustment of a figure prior to the computation of the area, where the area-formula assumes a closed mathematical figure.

Prior to the advent of programmable calculators and computers, Bowditch's Rule was often used to adjust traverses that did not close due to the effects of random errors in the measurement of bearings and distances. Its use was justified entirely by its simplicity and whilst it had theoretical rigour – if the bearings of traverse lines were independent of each other, as they are in compass surveys – it is incompatible with modern traversing techniques. Bowditch's rule cannot take into account different measurement precisions of individual traverse lines nor can it accommodate complicated networks of connecting traverses. Nevertheless, due to its long history of use in the surveying profession, its simplicity and its practical use in the computation of areas of figures that misclose, Bowditch's Rule is still prominent in surveying textbooks and is a useful adjustment technique.

Bowditch, Nathaniel (b. March 26, 1773, Salem, Mass., U.S. – d. March 16, 1838, Boston, Mass., U.S.), self-educated American mathematician and astronomer, author of the best book on navigation of his time, and discoverer of the **Bowditch** curves, which have important applications in astronomy and physics. Between 1795 and 1799 **Bowditch** made four lengthy sea voyages, and in 1802 he was put in command of a merchant vessel. Throughout that period he pursued his interest in mathematics. After investigating the accuracy of *The Practical Navigator,* a work by the Englishman J.H. Moore, he produced a revised edition in 1799. His additions became so numerous that in 1802 he published *The New American Practical Navigator,* based on Moore's book, which was adopted by the U.S. Department of the Navy and went through some 60 editions. **Bowditch** also wrote many scientific papers, one of which, on the motion of a pendulum swinging simultaneously about two axes at right angles, described the so-called **Bowditch** curves (better known as the Lissajous figures, after the man who later studied them in detail). **Bowditch** translated from the French and updated the first four volumes of Pierre-Simon Laplace's monumental work on the gravitation of heavenly bodies, *Traité de mécanique céleste,* more than doubling its size with his own commentaries. The resulting work, *Celestial Mechanics,* was published in four volumes in 1829-39. **Bowditch** refused professorships at several universities. He was president (1804-23) of the Essex Fire and Marine Insurance Company of Salem and worked as an actuary (1823-38) for the Massachusetts Hospital Life Insurance Company of Boston. From 1829 until his death, he was president of the American Academy of Arts and Sciences.

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[2] *The Analyst or Mathematical Museum* was a journal of theoretical and applied mathematics. In Vol. I, No. II, 1808, Robert Patterson of Philadelphia posed a question on the adjustment of a traverse and offered a prize of $10 for a solution; the editor Dr Adrian appointed as the judge of submissions. Bowditch's solution was published in Vol. I, No. IV, 1808, pp. 88-93 (Stoughton, H.W., 1974. 'The first method to adjust a traverse based on statistical considerations', *Surveying and Mapping,* June 1974, pp. 145-49).

Bowditch's adjustment can best be explained by considering the case of plotting a figure (using a protractor and scale ruler) given the bearing and distances of the sides.

Consider Figure 6.3, a plot that does not close, of a four-sided figure *ABCD*. The solid lines *AB*, *BC*, *CD* and *DE* are the result of marking point *A*, plotting the bearing *AB* and then scaling the distance *AB* to fix *B*. Then, from point *B*, plotting the bearing and distance *BC* to fix *C*, then from *C*, plotting the bearing and distance *CD* to fix *D* and finally from *D*, plotting the bearing and distance *DA*. However, due to plotting errors, the final line does not meet the starting point, but instead finishes at *E*. The distance *EA* is the linear misclose *d*, due to plotting errors, i.e., errors in protracting bearings and scaling distances.

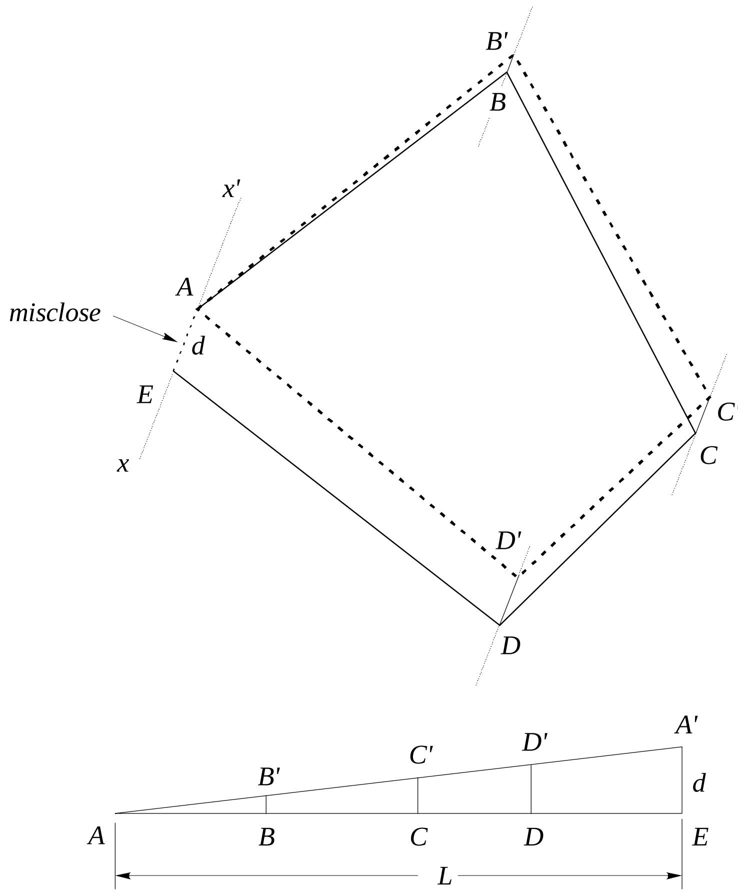

Figure 6.3 Graphical plot of polygon *ABCD* with misclose *d*

To adjust the figure *ABCDE* to remove the misclose *d* the following procedure can be used.

1. Draw lines parallel to the line *xx'* (the misclose bearing) through points *B, C* and *D.*
2. Draw a right-angled triangle *AEA'*. The base of the triangle is *L,* equal to the sum of the lengths of the sides and the height is the linear misclose *d.*
3. Along the base of the triangle, mark in proportion to the total length *L*, the distances *AB, BC* and *CD.* These will be the points *B, C* and *D.*
4. Draw vertical lines from *B, C* and *D* intersecting the hypotenuse of the triangle at *B', C'* and *D'.* These distances are then marked off along the parallel lines of the main figure.
5. The adjusted figure is *AB'C'D'A.*

This adjustment is a graphical demonstration of Bowditch's Rule; i.e., the linear misclose *d* is apportioned to individual sides in the ratio of the length of the side to the total length of all the sides in the direction of the misclose bearing.

Bowditch's Rule as it is normally applied to the adjustment of traverses can be deduced by again considering Figure 6.3. The linear misclose *d* has easting and northing components ΔE_m and ΔN_m, the subscript *m* referring to the misclose. The distances *BB', CC'* and *DD'* each have easting and northing components, say dE_B, dN_B, dE_C, dN_C and dE_D, dN_D, the east misclose $dE_m = dE_B + dE_C + dE_D$ and the north misclose $dN_m = dN_B + dN_C + dN_D$.

Thus, we may express Bowditch's Rule for calculating adjustments dE_k, dN_k to individual easting and northing components $\Delta E_k, \Delta N_k$ of line *k* of a traverse whose total length is *L* as

$$\begin{aligned} dE_k &= dist_k\left(\frac{dE_m}{L}\right) \\ dN_k &= dist_k\left(\frac{dN_m}{L}\right) \end{aligned} \tag{6.41}$$

As an example of a Bowditch adjustment, Table 6.1 shows the bearings and distances of the polygon in Figure 6.3.

The linear misclose, which is quite large, is $d = \sqrt{(-3.173)^2 + (-8.181)^2} = 8.775$ and the length *L,* equal to the sum of the four sides, is $L = 51.53 + 53.86 + 36.31 + 54.71 = 196.41$

The corrections to the easting and northing components of the line *CD* are

$$dE = 36.31 \times \frac{3.173}{196.41} = 0.587$$
$$dN = 36.31 \times \frac{8.181}{196.41} = 1.512$$

Note: (i) Easting and northing misclosures dE_m and dN_m used in equations (6.41) have opposite signs to the misclosures in the tabulation,

(ii) The sums of the corrections are equal and of opposite sign to the misclosures and

(iii) The sums of the adjusted easting and northing components are zero.

			components		*corrections*		*adjusted components*	
Line	*Bearing*	*Dist*	ΔE	ΔN	*dN*	*dN*	ΔE	ΔN
AB	52° 31'	51.53	40.891	31.358	0.832	2.146	41.723	33.504
BC	152° 21'	53.86	24.995	-47.709	0.870	2.243	25.865	-45.466
CD	225° 30'	36.31	-25.898	-25.450	0.587	1.512	-25.311	-23.938
DA	307° 55'	54.71	-43.161	33.620	0.884	2.280	-42.277	35.900
		misclose	-3.173	-8.181	3.173	8.181	0.000	0.000

Table 6.1. Bowditch Rule adjustment of polygon *ABCD*

Transit Rule

The Transit Rule has no theoretical basis related to surveying instruments or measuring techniques. Its only justification is its mathematical simplicity, which is no longer a valid argument for the method in this day of pocket computers. The Transit Rule for calculating adjustments dE_k, dN_k to individual easting and northing components $\Delta E_k, \Delta N_k$ of line *k* of a traverse whose east and north misclosures are dE_m and dN_m is

$$dE_k = |\Delta E_k| \left(\frac{dE_m}{\sum_{j=1}^{n} |\Delta E_j|} \right) \qquad dN_k = |\Delta N_k| \left(\frac{dN_m}{\sum_{j=1}^{n} |\Delta N_j|} \right) \tag{6.42}$$

$\left|\Delta E_k\right|$ is the absolute value of the east component of the k^{th} traverse leg and $\sum_{j=1}^{n}\left|\Delta E_j\right|$ is the sum of the absolute values of the east components of the traverse legs and similarly for $\left|\Delta N_k\right|$ and $\sum_{j=1}^{n}\left|\Delta N_j\right|$.

As an example of a Transit Rule adjustment, Table 6.2 shows the bearings and distances of the polygon in Figure 6.3. The east and north misclosures are $dE_m = 3.173$ and $dN_m = 8.181$, and the sums of the absolute values of the east and north components of the traverse legs are $\sum_{j=1}^{n}\left|\Delta E_j\right| = 134.945$ and $\sum_{j=1}^{n}\left|\Delta N_j\right| = 138.137$

The corrections to the easting and northing components of the line *CD* are

$$\begin{aligned} dE &= 25.898 \times \frac{3.173}{134.945} = 0.587 \\ dN &= 25.450 \times \frac{8.181}{138.137} = 1.512 \end{aligned}$$

Note: (i) Easting and northing misclosures dE_m and dN_m used in equations (6.42) have opposite signs to the misclosures in the tabulation,

(ii) The sums of the corrections are equal and of opposite sign to the misclosures and

(iii) The sums of the adjusted easting and northing components are zero.

			components		*corrections*		*adjusted components*	
Line	*Bearing*	*Dist*	ΔE	ΔN	*dN*	*dN*	ΔE	ΔN
AB	52° 31'	51.53	40.891	31.358	0.961	1.857	41.852	33.215
BC	152° 21'	53.86	24.995	-47.709	0.588	2.826	25.583	-44.883
CD	225° 30'	36.31	-25.898	-25.450	0.609	1.507	-25.289	-23.943
DA	307° 55'	54.71	-43.161	33.620	1.015	1.991	-42.146	35.611
		misclose	-3.173	-8.181	3.173	8.181	0.000	0.000

Table 6.2 Transit Rule adjustment of polygon *ABCD*

6.5.2. Crandall's method. A semi-rigorous single traverse adjustment method

Suppose that the angles of a traverse – either beginning and ending at the same point or between two known points with starting and closing known bearings – have been adjusted so that the traverse has a perfect angular closure and the resulting bearings are considered as correct, or adjusted. We call this a closed traverse. A mathematical closure, using the adjusted bearings and measured distances, will in all probability, reveal a linear misclose, i.e., the sums of the east and north components of the traverse legs will differ from zero (in the case of a traverse beginning and ending at the same point) or certain known values (in the case of a traverse between known points). *Crandall's method,* which employs the least squares principle, can be used to compute corrections to the measured distances to make the traverse close mathematically. The method was first set out in the textbook *Geodesy and Least Squares* by Charles L. Crandall, Professor of Railroad Engineering and Geodesy, Cornell University, Ithaca, New York, U.S.A. and published by John Wiley & Sons, New York, 1906.

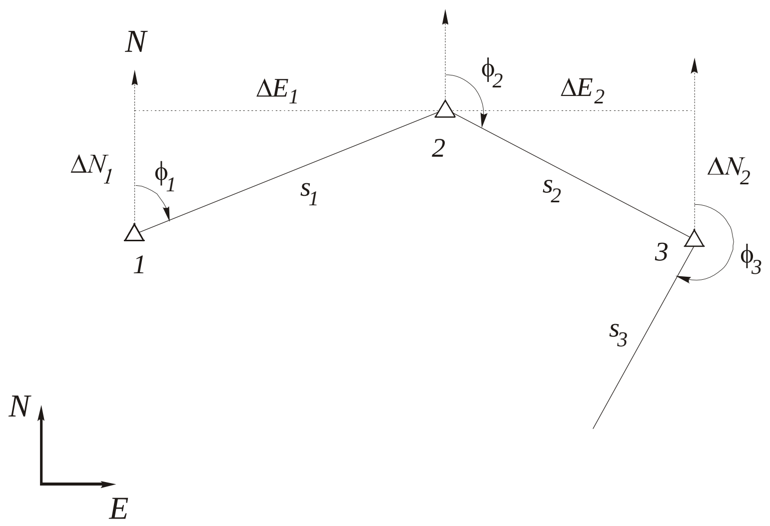

Figure 6.4 Schematic traverse diagram

Figure 6.4 shows a schematic diagram of a traverse of $k = 1, 2, \ldots, n$ legs where ϕ_k, s_k are the adjusted bearing and measured distance respectively of the k^{th} leg. The east and north components of each traverse leg are $\Delta E_k = s_k \sin\phi_k$ and $\Delta N_k = s_k \cos\phi_k$ respectively. If the adjusted distance of the k^{th} traverse leg is $(s_k + v_k)$ where v_k is the residual (a small unknown correction) then the two conditions that must be fulfilled by the adjusted bearings and adjusted distances in a closed traverse are

$$\begin{aligned}(s_1+v_1)\sin\phi_1 + (s_2+v_2)\sin\phi_2 + \cdots + (s_n+v_n)\sin\phi_n &= D_E \\ (s_1+v_1)\cos\phi_1 + (s_2+v_2)\cos\phi_2 + \cdots + (s_n+v_n)\cos\phi_n &= D_N\end{aligned} \tag{6.43}$$

where $D_E = E_{END} - E_{START}$ and $D_N = N_{END} - N_{START}$ are the east and north coordinate differences respectively between the terminal points of the traverse. Note that in a traverse beginning and ending at the same point D_E and D_N will both be zero.

Expanding equation (6.43) gives

$$\begin{aligned}v_1\sin\phi_1 + v_2\sin\phi_2 + \cdots + v_n\sin\phi_n + S_E &= D_E \\ v_1\cos\phi_1 + v_2\cos\phi_2 + \cdots + v_n\cos\phi_n + S_N &= D_N\end{aligned} \tag{6.44}$$

where

$$\begin{aligned}S_E &= s_1\sin\phi_1 + s_2\sin\phi_2 + \cdots + s_n\sin\phi_n = \sum_{k=1}^{n}\Delta E_k \\ S_N &= s_1\cos\phi_1 + s_2\cos\phi_2 + \cdots + s_n\cos\phi_n = \sum_{k=1}^{n}\Delta N_k\end{aligned} \tag{6.45}$$

S_E, S_N are the sums of the east and north components, $\Delta E_k, \Delta N_k$ respectively, of the $k = 1, 2, \ldots, n$ traverse legs.

Equations (6.44) can be expressed in matrix form as

$$\begin{bmatrix}\sin\phi_1 & \sin\phi_2 & \sin\phi_3 & \cdots & \sin\phi_n \\ \cos\phi_1 & \cos\phi_2 & \cos\phi_3 & \cdots & \cos\phi_n\end{bmatrix}\begin{bmatrix}v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n\end{bmatrix} = \begin{bmatrix}D_E - S_E \\ D_N - S_N\end{bmatrix} \tag{6.46}$$

or $$\mathbf{Av} = \mathbf{f}$$

The solution for the vector of residuals **v** is given by equations (6.24) and (6.26) re-stated again as

$$\begin{aligned}\mathbf{v} &= \mathbf{W}^{-1}\mathbf{A}^T\mathbf{k} = \mathbf{Q}\mathbf{A}^T\mathbf{k} \\ \mathbf{k} &= \left(\mathbf{A}\mathbf{Q}\mathbf{A}^T\right)^{-1}\mathbf{f}\end{aligned} \tag{6.47}$$

where **k** is the vector of *Lagrange multipliers,* $\mathbf{Q} = \mathbf{W}^{-1}$ is the cofactor matrix and **W** is the weight matrix, **A** is a coefficient matrix containing sines and cosines of traverse bearings and **f** is a vector containing the negative sums of the east and north components of the traverse legs.

In Crandall's method, weights are considered as inversely proportional to the measured distances and the measured distances are considered to be independent. Hence the weight matrix **W** is diagonal

$$\mathbf{W} = \begin{bmatrix} w_1 & 0 & 0 & \cdots & 0 \\ 0 & w_2 & 0 & \cdots & 0 \\ 0 & 0 & w_3 & & \vdots \\ \vdots & & & \ddots & \\ 0 & \cdots & & & w_n \end{bmatrix} = \begin{bmatrix} 1/s_1 & 0 & 0 & \cdots & 0 \\ 0 & 1/s_2 & 0 & \cdots & 0 \\ 0 & 0 & 1/s_3 & & \vdots \\ \vdots & & & \ddots & \\ 0 & \cdots & & & 1/s_n \end{bmatrix}$$

and since $\mathbf{Q} = \mathbf{W}^{-1}$ then

$$\mathbf{Q} = \begin{bmatrix} s_1 & 0 & 0 & \cdots & 0 \\ 0 & s_2 & 0 & \cdots & 0 \\ 0 & 0 & s_3 & & \vdots \\ \vdots & & & \ddots & \\ 0 & \cdots & & & s_n \end{bmatrix}$$

and

$$\mathbf{Q}\mathbf{A}^T = \begin{bmatrix} s_1 & 0 & 0 & \cdots & 0 \\ 0 & s_2 & 0 & \cdots & 0 \\ 0 & 0 & s_3 & & \vdots \\ \vdots & & & \ddots & \\ 0 & \cdots & & & s_n \end{bmatrix} \begin{bmatrix} \sin\phi_1 & \cos\phi_1 \\ \sin\phi_2 & \cos\phi_2 \\ \sin\phi_3 & \cos\phi_3 \\ \vdots & \vdots \\ \sin\phi_n & \cos\phi_n \end{bmatrix} = \begin{bmatrix} \Delta E_1 & \Delta N_1 \\ \Delta E_2 & \Delta N_2 \\ \Delta E_3 & \Delta N_3 \\ \vdots & \vdots \\ \Delta E_n & \Delta N_n \end{bmatrix} \tag{6.48}$$

$$\mathbf{A}\mathbf{Q}\mathbf{A}^T = \begin{bmatrix} \sum_{k=1}^{n} \sin\phi_k \Delta E_k & \sum_{k=1}^{n} \sin\phi_k \Delta N_k \\ \sum_{k=1}^{n} \cos\phi_k \Delta E_k & \sum_{k=1}^{n} \cos\phi_k \Delta N_k \end{bmatrix}$$

Now, since $\sin\phi_k = \dfrac{\Delta E_k}{s_k}$ and $\cos\phi_k = \dfrac{\Delta N_k}{s_k}$ then $\mathbf{AQA}^T$ can be written as

$$\mathbf{AQA}^T = \begin{bmatrix} \sum_{k=1}^{n} \dfrac{\left(\Delta E_k\right)^2}{s_k} & \sum_{k=1}^{n} \dfrac{\Delta E_k \Delta N_k}{s_k} \\ \sum_{k=1}^{n} \dfrac{\Delta E_k \Delta N_k}{s_k} & \sum_{k=1}^{n} \dfrac{\left(\Delta N_k\right)^2}{s_k} \end{bmatrix} = \begin{bmatrix} a & c \\ c & b \end{bmatrix} \tag{6.49}$$

and $$\left(\mathbf{AQA}^T\right)^{-1} = \frac{1}{ab - c^2}\begin{bmatrix} b & -c \\ -c & a \end{bmatrix}$$

giving the *Lagrange multipliers* from equations (6.47) as

$$\begin{aligned} k_1 &= \frac{b\left(D_E - S_E\right) - c\left(D_N - S_N\right)}{ab - c^2} \\ k_2 &= \frac{a\left(D_N - S_N\right) - c\left(D_E - S_E\right)}{ab - c^2} \end{aligned} \tag{6.50}$$

The residuals $\mathbf{v}$ (corrections to the measured distances) are given as

$$\mathbf{v} = \mathbf{QA}^T\mathbf{k} = \begin{bmatrix} \Delta E_1 & \Delta N_1 \\ \Delta E_2 & \Delta N_2 \\ \Delta E_3 & \Delta N_3 \\ \vdots & \vdots \\ \Delta E_n & \Delta N_n \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1\Delta E_1 + k_2\Delta N_1 \\ k_1\Delta E_2 + k_2\Delta N_2 \\ k_1\Delta E_3 + k_2\Delta N_3 \\ \vdots \\ k_1\Delta E_n + k_2\Delta N_n \end{bmatrix} \tag{6.51}$$

6.5.3. Example of Crandall's method

Figure 6.5 shows a closed traverse between stations *A, B, C, D* and *E*. The linear misclose (bearing and distance) of the traverse is 222º 57' 31" *0.2340* and the components of the misclose are -0.1594 m east and -0.1712 m north. It is required to adjust the distances using Crandall's method.

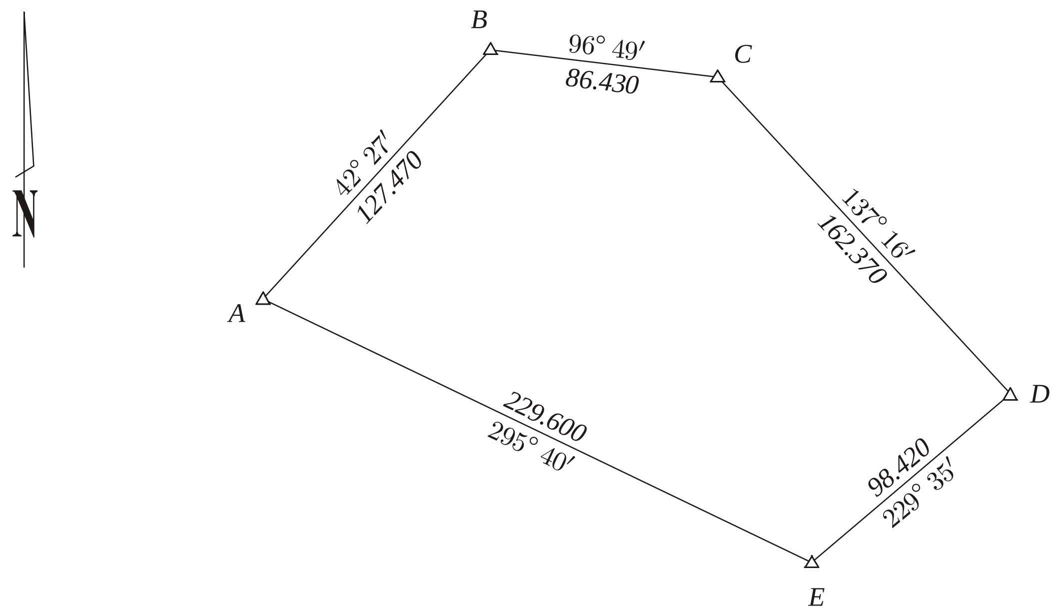

Figure 6.5 Closed traverse between stations *ABCDE*

The adjusted bearings and measured distances and the traverse leg components are shown in Table 6.3 below. S_E and S_N are the summations of east and north components and since this traverse begins and ends at the same point then D_E and D_N will both be zero.

Line	Bearing	Distance	traverse leg components	
k	ϕ_k	s_k	ΔE_k	ΔN_k
1	42º 27′	127.470	86.035437	94.055858
2	96º 49′	86.430	85.819028	-10.258619
3	137º 16′	162.370	110.182189	-119.264002
4	229º 35′	98.420	-74.932042	-63.809760
5	295º 40′	229.600	-206.945175	99.447747
			$S_E = 0.159438$	$S_N = 0.171224$

Table 6.3 Traverse components and sums

Table 6.4 shows the functions of the components for each line and their summations.

Line	$\frac{(\Delta E_k)^2}{s_k}$	$\frac{(\Delta N_k)^2}{s_k}$	$\frac{\Delta E_k \Delta N_k}{s_k}$
1	58.069322	69.400678	63.482677
2	85.212376	1.217624	-10.186101
3	74.768213	87.601787	-80.931014
4	57.049491	41.370509	48.581545
5	186.525721	43.074279	-89.635154
	$a = 461.625124$	$b = 242.664876$	$c =$ -68.688048

Table 6.4 Functions of traverse components

The Lagrange multipliers k_1 and k_2 are computed from equations (6.50) using *a, b, c* from Table 6.4, S_E and S_N from Table 6.3, and since this traverse begins and ends at the same point then D_E and D_N will both be zero.

$$a = \sum_{k=1}^{5} \frac{(\Delta E_k)^2}{s_k} = 461.625124 \qquad D_E - S_E = -0.159438 \qquad k_1 = -4.7018E-04$$

$$b = \sum_{k=1}^{5} \frac{(\Delta N_k)^2}{s_k} = 242.664876 \qquad D_N - S_N = -0.171224 \qquad k_2 = -8.3868E-04$$

$$c = \sum_{k=1}^{5} \frac{\Delta E_k \Delta N_k}{s_k} = -68.688048$$

Table 6.5 shows the original traverse data, the residuals and adjusted traverse distances.

Line	Bearing	Distance	Traverse leg components		Residual	Adjusted Distance
k	ϕ_k	s_k	ΔE_k	ΔN_k	$v_k = k_1 \Delta E_k + k_2 \Delta N_k$	
1	42° 27′	127.470	86.035437	94.055858	-0.119	127.351
2	96° 49′	86.430	85.819028	-10.258619	-0.032	86.398
3	137° 16′	162.370	110.182189	-119.264002	0.048	162.418
4	229° 35′	98.420	-74.932042	-63.809760	0.089	98.509
5	295° 40′	229.600	-206.945175	99.447747	0.014	229.614

Table 6.5 Adjusted traverse distances: Crandall's method

6.5.4. A rigorous single traverse adjustment method

A traverse is a combination of two basic survey measurements, *distances* and *directions*. Ignoring the physical fact that the same measuring equipment is likely to be used on each leg of the traverse, distances and directions are independently determined quantities. *Bearings* ϕ*,* *angles* α and *coordinates E,N* are derived quantities and in general, cannot be considered as mathematically (or statistically) independent. Restricting the adjustment method to single traverses, means angles at traverse points, derived from directions at those points, can be considered as mathematically independent quantities.

Three conditions, expressing the mathematical relationship between traverse measurements and derived coordinates, may be deduced from Figure 6.6 below, in which P_1 and P_n are "fixed stations" whose east and north coordinates are known and $P_2, P_3, P_4, \ldots, P_{n-1}$ are "floating stations" whose coordinates are to be determined from the traverse angles α and distances *s*. The starting bearing ϕ_0 and the finishing bearing ϕ_n are known.

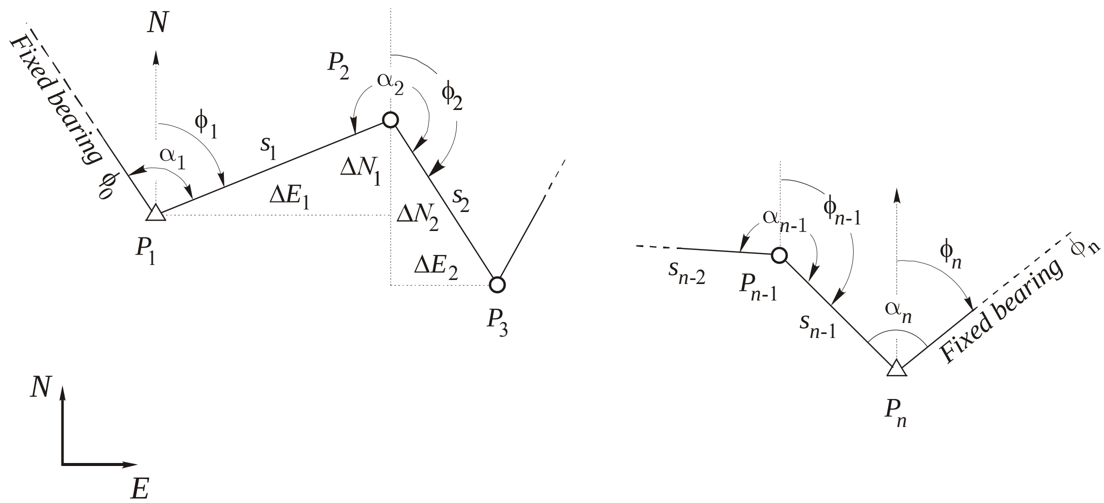

Figure 6.6 A closed traverse between two fixed stations

These three conditions are:

(i) The starting bearing ϕ_0 plus all the measured angles <u>should</u> equal the known finishing bearing ϕ_n,

(ii) The starting east coordinate plus all the east components of the traverse legs <u>should</u> equal the known east coordinate at the end point and

(iii) The starting north coordinate plus all the north components of the traverse legs <u>should</u> equal the known north coordinate at the end point.

<u>These conditions apply to all single traverses whether they start and end at different fixed points or close back on the starting point</u> and can be expressed mathematically as

$$\begin{aligned}
\phi_0 \ + \ \alpha_1 \ + \ \alpha_2 \ + \ \alpha_3 \ + \cdots + a_n \quad &= \phi_n \\
E_1 \ + \Delta E_1 + \Delta E_2 + \Delta E_3 + \cdots + \Delta E_{n-1} \ &= E_n \\
N_1 \ + \Delta N_1 + \Delta N_2 + \Delta N_3 + \cdots + \Delta N_{n-1} \ &= N_n
\end{aligned} \tag{6.52}$$

Equations (6.52) are relationships between <u>adjusted quantities</u> $\alpha_1, \alpha_2, \ldots, \alpha_n$ <u>or functions of adjusted quantities</u> $\Delta E_1, \Delta E_2, \ldots, \Delta E_{n-1}$ and $\Delta N_1, \Delta N_2, \ldots, \Delta N_{n-1}$.

Traverses will generally misclose due to the small random errors in the angles (derived from the measured directions) and the measured distances. To make the traverse mathematically correct, small corrections must be applied to the measurements to give adjusted quantities. These adjusted quantities are:

$$s = s' + v_s$$

$$\alpha = \alpha' + v_\alpha$$

where s and α are adjusted distance and angle respectively, s' and α' are the measured angle and distance, and v_s and v_α are small corrections. Replacing the adjusted quantities with measurements and corrections allows the first member of equations (6.52) to be expressed as

$$\phi_0 + \left(\alpha_1 + v_{\alpha_1}\right) + \left(\alpha_2 + v_{\alpha_2}\right) + \left(\alpha_3 + v_{\alpha_3}\right) + \cdots + \left(\alpha_n + v_{\alpha_n}\right) = \phi_n$$

and summing the measured angles and rearranging gives a simple expression for the summation of corrections to measured angles as

$$\boxed{v_{\alpha_1} + v_{\alpha_2} + v_{\alpha_3} + \cdots + v_{\alpha_n} = f_1} \tag{6.53}$$

where, apart from some multiple of 180º

$$\boxed{f_1 = \phi_n - \left(\phi_0 + \sum_{k=1}^{n} \alpha'_k \right) = \phi_n - \phi'_n} \tag{6.54}$$

f_1 is the angular misclose in the traverse and equation (6.54) simply states that the sum of the corrections to the measured angles is equal to the angular misclose.

The second and third members of equations (6.52) can also be expressed as a summation of corrections by considering the following

$$\Delta E = s \sin \phi \quad \text{and} \quad \Delta N = s \cos \phi$$

where $\Delta E, \Delta N$ are east and north components of a traverse leg and

$$s = s' + v_s \quad \text{and} \quad \phi = \phi' + v_\phi$$

where ϕ' and v_ϕ are "measured" bearing and correction respectively, hence we express the east and north components as

$$\begin{aligned} \Delta E &= \left(s' + v_s \right) \sin\left(\phi' + v_\phi \right) \\ \Delta N &= \left(s' + v_s \right) \cos\left(\phi' + v_\phi \right) \end{aligned}$$

Using the trigonometric expansions for $\sin\left(A+B\right)$ and $\cos\left(A+B\right)$, and the approximations $\sin v_\phi \simeq v_\phi$ and $\cos v_\phi \simeq 1$ since v_ϕ is a <u>small</u> quantity gives

$$\begin{aligned} \Delta E &= \left(s' + v_s \right) \left\{ \sin\phi' \cos v_\phi + \sin v_\phi \cos\phi' \right\} = s'\sin\phi' + s'v_\phi \cos\phi' + v_s \sin\phi' + v_s v_\phi \cos\phi' \\ \Delta N &= \left(s' + v_s \right) \left\{ \cos\phi' \cos v_\phi - \sin\phi' \sin v_\phi \right\} = s'\cos\phi' - s'v_\phi \sin\phi' + v_s \cos\phi' - v_s v_\phi \sin\phi' \end{aligned}$$

and since v_s and v_ϕ are <u>both small</u> then their product $v_s v_\phi \simeq 0$, hence

$$\begin{aligned} \Delta E &= s'\sin\phi' + v_\phi s'\cos\phi' + v_s \sin\phi' \\ \Delta N &= s'\cos\phi' - v_\phi s'\sin\phi + v_s \cos\phi' \end{aligned}$$

Finally, the east and north components of a traverse leg computed using the measured quantities are $\Delta E' = s'\sin\phi'$ and $\Delta N' = s'\cos\phi'$, and we may write

$$\begin{aligned}\Delta E &= \Delta E' + v_\phi \Delta N' + v_s \sin\phi' \\ \Delta N &= \Delta N' - v_\phi \Delta E' + v_s \cos\phi'\end{aligned} \tag{6.55}$$

Substituting equations (6.55) into the second and third members of equations (6.52) gives

$$\begin{aligned}E_1 \quad &+\left(\Delta E_1{}' + v_{\phi_1}\Delta N_1' + v_{s_1}\sin\phi_1'\right)\\ &+\left(\Delta E_2{}' + v_{\phi_2}\Delta N_2' + v_{s_2}\sin\phi_2'\right)\\ &+\cdots\\ &+\left(\Delta E_{n-1}' + v_{\phi_{n-1}}\Delta N_{n-1}' + v_{s_{n-1}}\sin\phi_{n-1}'\right) = E_n\\ N_1 \quad &+\left(\Delta N_1{}' - v_{\phi_1}\Delta E_1' + v_{s_1}\cos\phi_1'\right)\\ &+\left(\Delta N_2' - v_{\phi_2}\Delta E_2' + v_{s_2}\cos\phi_2'\right)\\ &+\cdots\\ &+\left(\Delta N_{n-1}' - v_{\phi_{n-1}}\Delta E_{n-1}' + v_{s_{n-1}}\cos\phi_{n-1}'\right) = N_n\end{aligned}$$

Letting the misclose in the east and north coordinates be

$$\boxed{\begin{aligned}f_2 &= E_n - \left\{E_1 + \sum_{k=1}^{n-1}\Delta E_k'\right\} = E_n - E_n'\\ f_3 &= N_n - \left\{N_1 + \sum_{k=1}^{n-1}\Delta N_k'\right\} = N_n - N_n'\end{aligned}} \tag{6.56}$$

and recognising that $v_{\phi_1} = v_{\alpha_1}$, $v_{\phi_2} = v_{\alpha_1} + v_{\alpha_2}$, $v_{\phi_3} = v_{\alpha_1} + v_{\alpha_2} + v_{\alpha_3}$ etc, and $v_{\phi_{n-1}} = \sum_{k=1}^{n-1} v_{\alpha_k}$ then we may write

$$\begin{aligned}&v_{\alpha_1}\Delta N_1' + v_{s_1}\sin\phi_1' + \left(v_{\alpha_1} + v_{\alpha_2}\right)\Delta N_2' + v_{s_2}\sin\phi_2' + \left(v_{\alpha_1} + v_{\alpha_2} + v_{\alpha_3}\right)\Delta N_3' + v_{s_3}\sin\phi_3' + \cdots\\ &\cdots + \left(\sum_{k-1}^{n-1} v_{\alpha_k}\right)\Delta N_{n-1}' + v_{s_{n-1}}\sin\phi_{n-1}' = f_2\end{aligned}$$

$$\begin{aligned}&-v_{\alpha_1}\Delta E_1' + v_{s_1}\cos\phi_1' - \left(v_{\alpha_1} + v_{\alpha_2}\right)\Delta E_2' + v_{s_2}\cos\phi_2' - \left(v_{\alpha_1} + v_{\alpha_2} + v_{\alpha_3}\right)\Delta E_3' + v_{s_3}\cos\phi_3' - \cdots\\ &\cdots - \left(\sum_{k-1}^{n-1} v_{\alpha_k}\right)\Delta E_{n-1}' + v_{s_{n-1}}\cos\phi_{n-1}' = f_3\end{aligned}$$

Gathering together the coefficients of $v_{\alpha_1}, v_{\alpha_2}, v_{\alpha_3},$ etc and rearranging gives

$$\boxed{\begin{aligned}&\left(\sin\phi_1'\right)v_{s_1} + \left(\sin\phi_2'\right)v_{s_2} + \left(\sin\phi_3'\right)v_{s_3} + \cdots + \left(\sin\phi_{n-1}'\right)v_{s_{n-1}}\\ &+\left(N_n' - N_1\right)v_{\alpha_1} + \left(N_n' - N_2'\right)v_{\alpha_2} + \left(N_n' - N_3'\right)v_{\alpha_3} + \cdots + \left(N_n' - N_{n-1}'\right)v_{\alpha_{n-1}} = f_2\end{aligned}} \tag{6.57}$$

$$\boxed{\begin{aligned}&\left(\cos\phi_1'\right)v_{s_1}+\left(\cos\phi_2'\right)v_{s_2}+\left(\cos\phi_3'\right)v_{s_3}+\cdots+\left(\cos\phi_{n-1}'\right)v_{s_{n-1}}\\&-\left(E_n'-E_1\right)v_{\alpha_1}-\left(E_n'-E_2'\right)v_{\alpha_2}-\left(E_n'-E_3'\right)v_{\alpha_3}-\cdots-\left(E_n'-E_{n-1}'\right)v_{\alpha_{n-1}}=f_3\end{aligned}} \qquad (6.58)$$

Equations (6.53), (6.57) and (6.58) are the three equations that relate corrections to angles and distances, v_α and v_s respectively to angular and coordinate misclosures f_1, f_2 and f_3 given by equations (6.54) and (6.56). In equation (6.53) the coefficients of corrections to angles are all unity, whilst in equations (6.57) and (6.58) the coefficients of the corrections are sines and cosines of bearings and coordinate differences derived from the measurements. Equations (6.53), (6.57) and (6.58) are applicable to any single closed traverse.

6.5.5. Application of Least Squares Adjustment of Observations Only to Particular Single Closed Traverses

There are three types of single closed traverses.

Type I Traverses that begin and end at different fixed points with fixed orienting bearings at the terminal points. Figure 6.7(a).

Type II Traverses that begin and end at the same point with a single fixed orienting bearing. Figure 6.7(b)

Type III Traverses that begin and end at the same point with a fixed datum bearing. Figure 6.7(b)

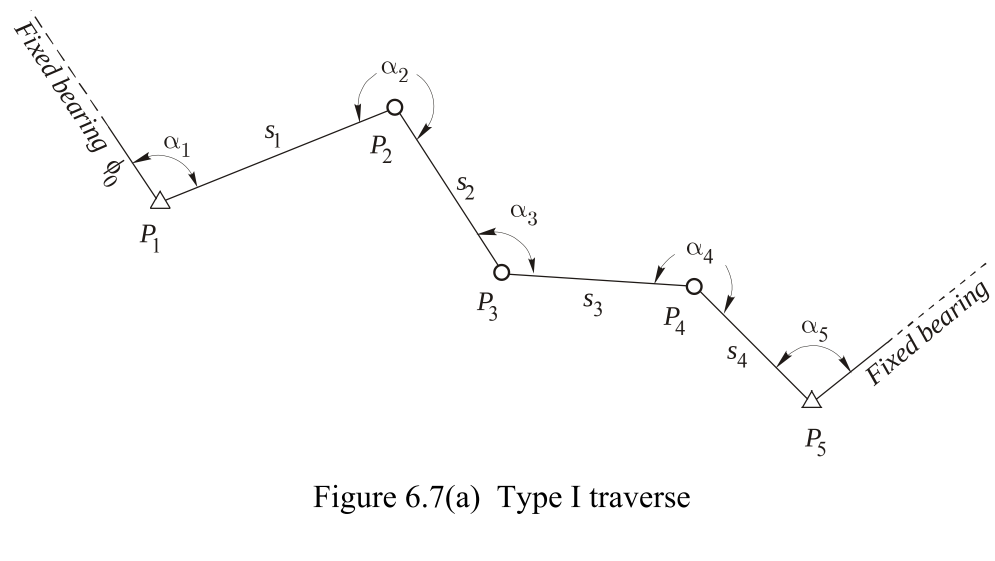

Figure 6.7(a) Type I traverse

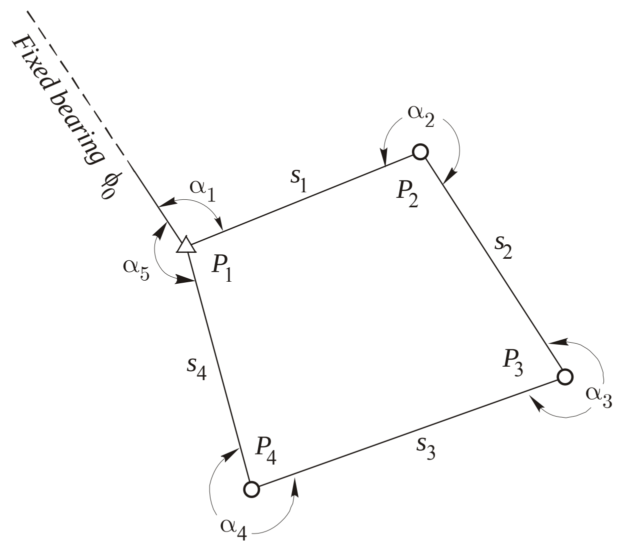

Figure 6.7(b) Type II traverse

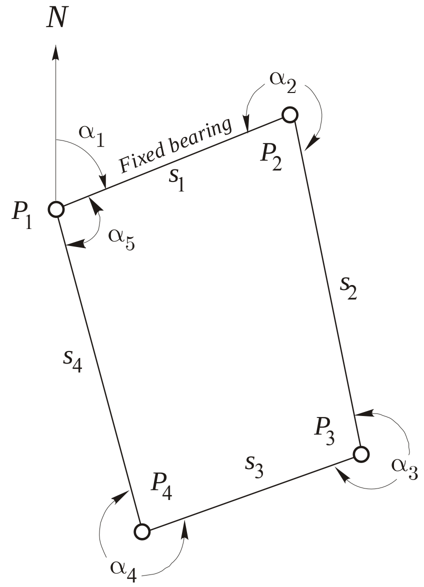

Figure 6.7(c) Type III traverse

Figures 6.7(a), 6.7(b) and 6.7(c) show three types of closed traverses. In each case, the traverse consists of four(4) distances s_1 to s_4 and five(5) angles α_1 to α_5. Traverse points shown with a triangle (Δ) can be regarded as fixed with known coordinates.

In Figures 6.7(a) and 6.7(b) the bearing of the traverse line $P_1 \rightarrow P_2$ is found by adding the observed angle α_1 to the fixed bearing ϕ_0. In both of these traverses five angles must be observed to "close" the traverse.

In Figure 6.7(c) the bearing of the traverse line $P_1 \rightarrow P_2$ is fixed and only four angles need be observed to close the traverse. The angle α_1 at P_1, clockwise from north to P_2, is the bearing of the traverse line $P_1 \rightarrow P_2$. α_1 is used in the adjustment as an observation with a standard deviation of zero.

For any single closed traverse, the method of adjustment is as follows:

(i) Calculate the coordinates of the traverse points by using the observed bearings and distances beginning at point P_1.

(ii) Calculate the angular and coordinate misclosures. In each case, the misclose is the fixed value minus the observed or calculated value. These three values are the elements f_1, f_2 and f_3 in the vector of numeric terms **f**

(iii) Calculate the coefficients of the correction (or residuals) in equations (6.53), (6.57) and (6.58). These coefficients are either zero or unity for equation (6.53), or sines and cosines of observed bearings together with coordinate differences in equations (6.57) and (6.58). These values are the elements of the coefficient matrix **A**

(iv) Assign precisions (estimated standard deviations squared) of the observations. These will be the diagonal elements of the cofactor matrix **Q**

Note: In Type III traverses where the bearing $P_1 \rightarrow P_2$ is fixed, the angle α_1 (which is not observed) is assigned a variance (standard deviation squared) of zero.

(v) Form a set of three(3) normal equations $\left(\mathbf{AQA}^T\right)\mathbf{k} = \mathbf{f}$

(vi) Solve the normal equations for the three(3) Lagrange multipliers k_1, k_2 and k_3 which are the elements of the vector **k** from $\mathbf{k} = \left(\mathbf{AQA}^T\right)^{-1}\mathbf{f}$ and then compute the vector of residuals (corrections) from $\mathbf{v} = \mathbf{QA}^T\mathbf{k}$

(vii) Calculate the adjusted bearings and distances of the traverse by adding the corrections to the observed angles and distances.

6.5.6. Example of Traverse Adjustment using Least Squares Adjustment of Observations Only

Figure 6.8 is a schematic diagram of a traverse run between two fixed stations *A* and *B* and oriented at both ends by angular observations to a third fixed station *C*.

The bearings of traverse lines shown on the diagram, unless otherwise indicated, are called "observed" bearings and have been derived from the measured angles (which have been derived from observed theodolite directions) and the fixed bearing *AC*. The difference between the observed and fixed bearings of the line *BC* represents the angular misclose. The coordinates of the traverse points *D, E* and *F* have been calculated using the observed bearings and distances and the fixed coordinates of *A*. The difference between the observed and fixed coordinates at *B* represents the coordinate misclosures.

In this example estimated standard deviations of measured angles α are $s_\alpha = 5''$ and for measured distances *s* are $s_s = 10\text{mm} + 15\text{ppm}$ where ppm is parts per million.

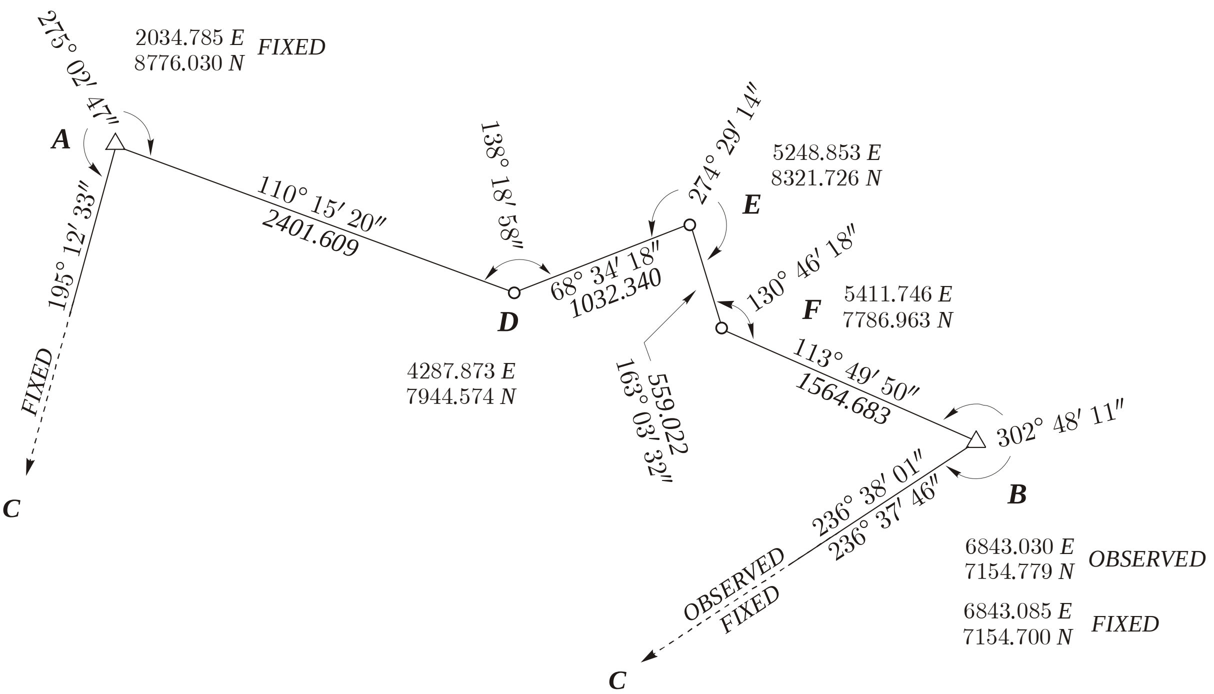

Figure 6.8 Traverse diagram showing field measurements, derived values and fixed values.

Step 1: Calculation of misclosures and formation of vector **f**

From equations (6.54) and (6.56) the angular and coordinate misclosures are the elements f_1, f_2 and f_3 of the vector of numeric terms **f**. These misclosures may be characterised as *misclose = fixed - observed*

angular misclose:
$$\begin{aligned} f_1 &= \phi_n - \phi'_n \\ &= 236^\circ\, 37'\, 46'' - 236^\circ\, 38'\, 01'' \\ &= -15'' \end{aligned}$$

east misclose:
$$\begin{aligned} f_2 &= E_n - E'_n \\ &= 6843.085 - 6843.030 \\ &= 0.055 \text{ m} \\ &= 5.5 \text{ cm} \end{aligned}$$

north misclose:
$$\begin{aligned} f_3 &= N_n - N'_n \\ &= 7154.700 - 7154.779 \\ &= -0.079 \text{ m} \\ &= -7.9 \text{ cm} \end{aligned}$$

vector of numeric terms:
$$\mathbf{f} = \begin{bmatrix} -15 \\ 5.50 \\ -7.9 \end{bmatrix} \begin{matrix} \text{sec} \\ \text{cm} \\ \text{cm} \end{matrix}$$

Note that the units of the numeric terms are seconds of arc (sec) and centimetres (cm)

Step 2: Form the coefficient matrix **A** of the equations (6.16) **Av = f**

The first row of **A** contains coefficients of zero or unity from equation (6.53)

$$v_{\alpha_1} + v_{\alpha_2} + v_{\alpha_3} + \cdots + v_{\alpha_n} = f_1$$

The second row of **A** contains the coefficients $\sin\phi'_k$ and $\left(N'_n - N'_k\right)\left(\dfrac{100}{\rho''}\right)$ from equation (6.57).

$$\begin{aligned} &\left(\sin\phi'_1\right)v_{s_1} + \left(\sin\phi'_2\right)v_{s_2} + \left(\sin\phi'_3\right)v_{s_3} + \cdots + \left(\sin\phi'_{n-1}\right)v_{s_{n-1}} \\ &+\left(N'_n - N_1\right)v_{\alpha_1} + \left(N'_n - N'_2\right)v_{\alpha_2} + \left(N'_n - N'_3\right)v_{\alpha_3} + \cdots + \left(N'_n - N'_{n-1}\right)v_{\alpha_{n-1}} = f_2 \end{aligned}$$

Note that the coefficients of the distance residuals are dimensionless quantities and the coefficients of the angle residuals have the dimensions of sec/cm where $\rho'' = \frac{180}{\pi} \times 3600$ is the number of seconds in one radian.

The third row of **A** contains the coefficients $\cos\phi_k'$ and $-\left(E_n' - E_k'\right)\left(\frac{100}{\rho''}\right)$ from equation (6.58).

$$\begin{aligned}&\left(\cos\phi_1'\right)v_{s_1} + \left(\cos\phi_2'\right)v_{s_2} + \left(\cos\phi_3'\right)v_{s_3} + \cdots + \left(\cos\phi_{n-1}'\right)v_{s_{n-1}} \\ &-\left(E_n' - E_1\right)v_{\alpha_1} - \left(E_n' - E_2'\right)v_{\alpha_2} - \left(E_n' - E_3'\right)v_{\alpha_3} - \cdots - \left(E_n' - E_{n-1}'\right)v_{\alpha_{n-1}} = f_3\end{aligned}$$

Note that the coefficients of the distance residuals are dimensionless quantities and the coefficients of the angle residuals have the dimensions of sec/cm where $\rho'' = \frac{180}{\pi} \times 3600$ is the number of seconds in one radian. The equation $\mathbf{Av} = \mathbf{f}$ is

$$\begin{array}{c} \left[\begin{array}{cccc:ccccc} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0.9382 & 0.9309 & 0.2914 & 0.9147 & -0.7860 & -0.3829 & -0.5658 & -0.3065 & 0 \\ -0.3462 & 0.3653 & -0.9566 & -0.4040 & -2.3311 & -1.2388 & -0.7729 & -0.6939 & 0 \end{array}\right] \\ \begin{array}{cccc:ccccc} 4.6\text{ cm} & 2.5\text{ cm} & 1.8\text{ cm} & 3.3\text{ cm} & 5'' & 5'' & 5'' & 5'' & 5'' \end{array} \end{array} \begin{array}{c} \left[\begin{array}{c} v \\ v \\ v \\ v \\ \hline v \\ v \\ v \\ v \\ v \end{array}\right] \end{array} \begin{array}{c} \uparrow \\ \text{angles} \\ \\ \downarrow \\ \hline \uparrow \\ \\ \text{distances} \\ \\ \downarrow \end{array} = \left[\begin{array}{c} -15 \\ 5.50 \\ -7.90 \end{array}\right]$$

Note that the numbers below the columns of **A** are the estimates of the standard deviations of the distances or angles associated with the coefficients.

Step 3: <u>Form the normal equations</u>

The normal equations are given by equations (6.25) as $\left(\mathbf{AQA}^T\right)\mathbf{k} = \mathbf{f}$

where $\mathbf{Q} = \mathbf{W}^{-1}$ is the cofactor matrix containing estimates of the variances of the measurements. **Q** and the weight matrix are **W** are square diagonal matrices, i.e., all off-diagonal elements are zero and since weights are inversely proportional to the estimates of the variances, the diagonal elements of $\mathbf{Q} = \left\{ \frac{1}{s_{\alpha 1}^2} \quad \frac{1}{s_{\alpha 2}^2} \quad \frac{1}{s_{\alpha 3}^2} \quad \frac{1}{s_{\alpha 4}^2} \;\vdots\; \frac{1}{s_{s1}^2} \quad \frac{1}{s_{s2}^2} \quad \frac{1}{s_{s3}^2} \quad \frac{1}{s_{s4}^2} \quad \frac{1}{s_{s5}^2} \right\}$ where the first 4 elements relate to the angles and the remaining 5 elements relate to the distances. Now consider a diagonal matrix that denoted $\sqrt{\mathbf{Q}}$ whose diagonal elements are the square-roots of the elements of $\mathbf{Q}$ and $\mathbf{Q} = \sqrt{\mathbf{Q}}\sqrt{\mathbf{Q}}$ and another matrix $\overline{\mathbf{A}} = \mathbf{A}\sqrt{\mathbf{Q}}$. Each element of $\overline{\mathbf{A}}$ is the original element of **A** multiplied by the estimate of the standard deviation associated with the particular element and the normal equations are given by $\left(\overline{\mathbf{A}}\overline{\mathbf{A}}^T\right)\mathbf{k} = \mathbf{f}$ where

$$\overline{\mathbf{A}} = \left[\begin{array}{cccc:ccccc} 0 & 0 & 0 & 0 & 5 & 5 & 5 & 5 & 5 \\ 4.3155 & 2.3272 & 0.5245 & 3.0186 & -1.9145 & -1.9145 & -2.8288 & -1.5324 & 0 \\ -1.5926 & 0.9134 & -1.7219 & -1.3333 & -6.1939 & -6.1939 & -3.8644 & -3.4696 & 0 \end{array}\right]$$

and
$$\overline{\mathbf{A}}\overline{\mathbf{A}}^T = \begin{bmatrix} 125 & -51.0285 & -125.9168 \\ -51.0285 & 62.8875 & 64.2375 \\ -125.9168 & 64.2375 & 209.2995 \end{bmatrix}$$

Step 4: <u>Solve the normal equations for the vector of Lagrange multipliers **k**</u>

From equations (6.47) and with the modification mentioned above

$$\mathbf{k} = \left(\mathbf{AQA}^T\right)^{-1}\mathbf{f} = \left(\overline{\mathbf{A}}\overline{\mathbf{A}}^T\right)^{-1}\mathbf{f} = \begin{bmatrix} 0.0219 & 0.0063 & 0.0112 \\ & 0.0250 & -0.0039 \\ \text{symmetric} & & 0.0127 \end{bmatrix}\begin{bmatrix} -15 \\ 5.5 \\ -7.9 \end{bmatrix}$$

and
$$\mathbf{k} = \begin{bmatrix} -0.3825 \\ 0.0738 \\ -0.2906 \end{bmatrix}$$

Step 5: Calculation of residuals and adjusted traverse dimensions

The residuals are obtained from equation (6.24) $\mathbf{v} = \mathbf{Q}\mathbf{A}^T\mathbf{k}$

Since the cofactor matrix **Q** is diagonal, the individual residuals can be calculated from

$$v_j = s_j^2\left(a_{1j}k_1 + a_{2j}k_2 + a_{3j}k_3\right) \tag{6.59}$$

where

$a_{1j},\ a_{2j},\ a_{3j}$	are elements of the coefficient matrix **A**
$k_1,\ k_2,\ k_3$	are the elements of the vector **k**
s_j^2	is the estimate of the variance of the j^{th} measurement

For example, the residual for the second distance $(j = 2)$ is

$$(2.5)^2\left\{(0)(-0.3825) + (0.9309)(0.0738) + (0.3653)(-0.2906)\right\} = -0.23 \text{ cm}$$

and the residual for the third measured angle $(j = 7)$ is

$$(5)^2\left\{(1)(-0.3825) + (-0.5658)(0.0738) + (-0.7729)(-0.2906)\right\} = -4.99''$$

Exactly the same result can be obtained by using the estimate of the standard deviations s_j and the elements of the matrix $\overline{\mathbf{A}}$

$$v_j = s_j\left(\bar{a}_{1j}k_1 + \bar{a}_{2j}k_2 + \bar{a}_{3j}k_3\right) \tag{6.60}$$

Both methods give

$$\mathbf{v} = \begin{bmatrix} 3.59 \text{ cm} \\ -0.23 \\ 0.97 \\ 2.01 \text{ cm} \\ 5.92'' \\ -1.27 \\ -4.99 \\ -5.09 \\ -9.56'' \end{bmatrix} \begin{array}{c} \uparrow \\ \\ \text{distances} \\ \downarrow \\ \hline \uparrow \\ \\ \text{angles} \\ \\ \downarrow \end{array}$$

The residuals for the bearings are the cumulative residuals for the angles up to the particular traverse line. They are

$$\mathbf{v}_{\phi} = \begin{bmatrix} 5.92'' \\ 4.65 \\ -0.34 \\ -5.43 \\ -14.99'' \end{bmatrix}$$

Applying these residuals (or corrections) to the measured quantities gives the adjusted traverse dimensions as

Line	Bearing	Distance
k	ϕ_k	s_k
1	110º 27′ 25.9″	2401.645
2	68º 34′ 22.6″	1032.338
3	163º 03′ 31.7″	559.032
4	113º 49′ 44.6″	1564.703

Table 6.6 Adjusted traverse distances

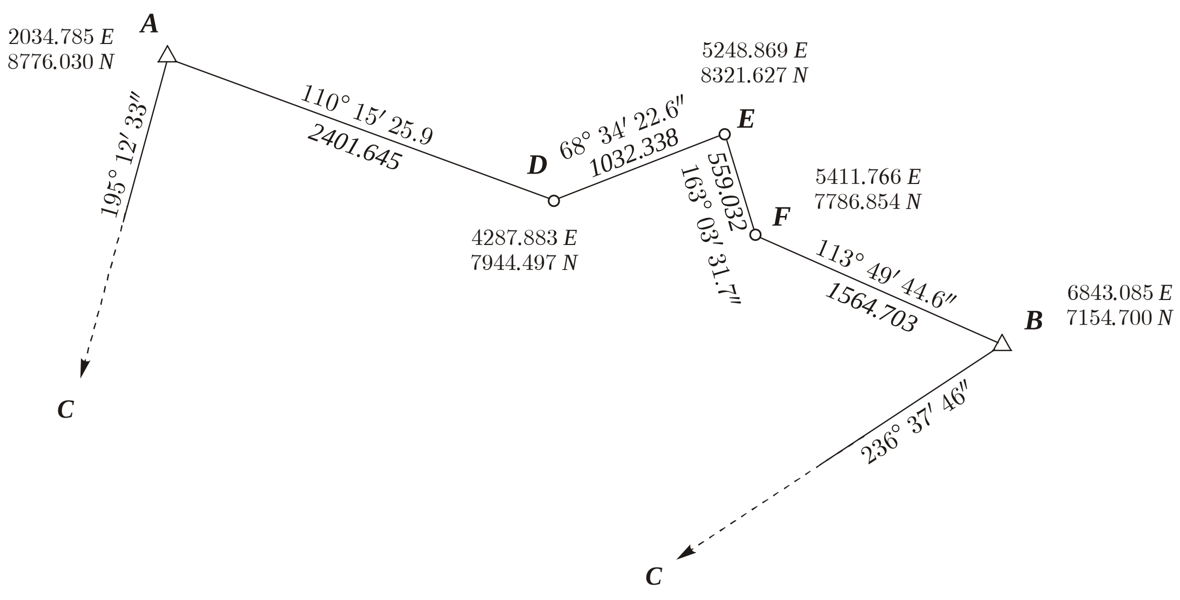

Figure 6.9 Traverse diagram showing adjusted measurements.